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An asymptotic solution of the tidal equations

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The Cauchy problem for the β-plane form of the tidal equations is solved for both oscillatory and delta function initial data. The radius of deformation is assumed to be much less than the radius of the earth, and in accord with this assumption a ray approximation is employed.

It is shown that, owing to the rapid rate of propagation of inertio-gravity waves, the motion in its initial development tends towards geostrophic balance. However, the solution given by the ray approximation is singular on certain surfaces in space and time, the envelopes of the rays. A local boundary-layer theory is employed to correct this deficiency. The existence of these caustics implies that the process of geostrophic adjustment is more complicated than hitherto imagined.

1. Introduction

One possible explanation for the well-known fact that gravity waves play a unimportant role in the large-scale motion of the atmosphere is that these waves, because of their high rate of propagation, spread out rapidly from local sources, leaving the slower Rossby waves behind. This process is called geostrophic adjustment. In the standard treatment (Obukhov 1949) a flat earth idealization of the tidal equations is solved as an initial value problem, and the motion indeed tends towards geostrophic balance as time $\rightarrow \infty$.

Among the defects of this model is the neglect of curvature effects and the consequent omission of Rossby waves and of refraction of the gravity waves. Accordingly, there is some interest in a treatment in which the effect of the earth's curvature is not totally neglected.

In this paper we consider the β-plane model and thus allow for the above-mentioned omissions of Obukhov's study. A parameter N, the ratio of the radius of the earth to the radius of deformation, is assumed to be large, and an approximate solution is found on this basis. The approximation is similar to the ray method used by Keller (1958) for the treatment of diffraction problems, though here it is used to solve an initial value problem.

As in diffraction theory the ray approximation is invalid on envelopes of the rays. A local boundary-layer theory is needed to correct this deficiency and is supplied in part. Near and on the envelopes the amplitudes of the dependent variables are large. This indicates that the process of geostrophic adjustment is considerably more complicated than would appear from a study of the flat earth model.

2. Formulation

Let λ and θ measure longitude and latitude on an earth of radius R rotating about a polar axis with angular velocity Ω, and let g be the gravitational constant. It is assumed that $\Omega^2 R/g$ is small so that ellipticity of geopotential surfaces can be neglected. We consider here a homogeneous fluid of uniform depth H with a free surface, but bear in mind that the theory applies equally well to inhomogeneous fluids provided H is replaced by an appropriate scale height.

Let ζ be the surface elevation, u and v velocity components to the east and north, and let x and y be Mercator co-ordinates defined by

$$x = \lambda, \quad y = \log\left(\frac{1+\sin\theta}{\cos\theta}\right). \tag{1}$$

We ignore the tide-generating forces. Then, in accord with the usual approximations of tidal theory (Lamb 1932, chapter 8), the equations of motion are

$$\frac{\partial u}{\partial t} - 2\Omega f v + \frac{gm}{R}\frac{\partial \zeta}{\partial x} = 0, \tag{2}$$

$$\frac{\partial v}{\partial t} + 2\Omega f u + \frac{gm}{R}\frac{\partial \zeta}{\partial y} = 0, \tag{3}$$

$$\frac{\partial \zeta}{\partial t} + \frac{gH}{R} m^2 \left[\frac{\partial}{\partial x}(u/m) + \frac{\partial}{\partial y}(v/m)\right] = 0, \tag{4}$$

where

$$f = \sin\theta = \tanh y, \quad m = \sec\theta = \cosh y. \tag{5}$$

In the β-plane approximation used here we replace m by unity and f by y. Introducing this approximation, scaling the variables through

$$\left.\begin{aligned} \zeta &= h\zeta^*, \\ (u,v) &= h(g/H)^{\frac{1}{2}}(u^*,v^*), \\ (x,y) &= (x^*,y^*), \\ t &= R(gH)^{-\frac{1}{2}}t^*, \end{aligned}\right\} \tag{6}$$

where h is a characteristic amplitude of the surface elevation, letting

$$N = 2\Omega R/(gH)^{\frac{1}{2}}, \tag{7}$$

and omitting the asterisks, we obtain

$$\frac{\partial u}{\partial t} - Nyv + \frac{\partial \zeta}{\partial x} = 0, \tag{8}$$

$$\frac{\partial v}{\partial t} + Nyu + \frac{\partial \zeta}{\partial y} = 0, \tag{9}$$

and

$$\frac{\partial \zeta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{10}$$

These are to be solved subject to initial conditions and to the requirements that the dependent variables be bounded at $|y| = \infty$ and periodic in x with period 2π.

It proves convenient to cast the problem in terms of v alone. Elimination of ζ between (8) and (10) and between (9) and (10) and of u between the resulting equations yields

$$\nabla^2 \frac{\partial v}{\partial t} - \frac{\partial^3 v}{\partial t^3} + N\frac{\partial v}{\partial x} - N^2 y^2 \frac{\partial v}{\partial t} = 0. \tag{11}$$

Similarly, the initial data become

$$\left.\begin{aligned} v = \tilde{v}, \quad v_t = -Ny\tilde{u} - \tilde{\zeta}_y, \\ v_{tt} = Ny(\tilde{\zeta}_x - Ny\tilde{v}) + \tilde{u}_{xy} + \tilde{v}_{yy}, \end{aligned}\right\} \tag{12}$$

where a tilde over a variable denotes its value at $t = 0$.

We will suppose that N is large and exploit this in seeking an asymptotic solution. Later, in §5, we shall consider the initial data

$$\tilde{u} = \tilde{v} = 0, \quad \tilde{\zeta} = \delta(x - x_0)\,\delta(y - y_0). \tag{13}$$

Here, however, we treat the case

$$\tilde{u} = \tilde{v} = 0, \quad \tilde{\zeta} = A(x, y)\exp\{iNkx\}, \tag{14}$$

where k is a constant of order unity and where

$$A(x + 2\pi, y) = \exp\{2\pi iNk\}\,A(x, y)$$

in agreement with the sentence following (10). This initial disturbance is in the form of a wave with variable amplitude and whose wavelength is short compared with the radius of the earth.

3. The ray approximation

We assume that the asymptotic expansion of v is of the form

$$v = w_R(x, y, \tau, N)\exp\{iN\phi_R(x, y, \tau)\} + w_+(x, y, t, N)\exp\{iN\phi_+(x, y, t)\} + w_-(x, y, t, N)\exp\{iN\phi_-(x, y, t)\}, \tag{15}$$

where

$$\tau = N^{-1}t, \tag{16}$$

where the first term of the sum is to satisfy

$$\left(N^2\nabla^2 \frac{\partial}{\partial \tau} - \frac{\partial^3}{\partial \tau^3} + N^4 \frac{\partial}{\partial x} - N^4 y^2\right) w_R \exp\{iN\phi_R\} = 0 \tag{17}$$

and where each of the last two terms satisfies

$$\left(\nabla^2 \frac{\partial}{\partial t} - \frac{\partial^3}{\partial t^3} + N\frac{\partial}{\partial x} - N^2 y^2 \frac{\partial}{\partial t}\right) w_\pm \exp\{iN\phi_\pm\} = 0. \tag{18}$$

The motivation for this assumption is due in equal measure to known facts about long wave motions in the atmosphere and oceans and to what is now classical singular perturbation theory.

We note first that there are two classes of long waves which are often observed in geophysical contexts, the inertio-gravity and the Rossby waves. In elementary treatments their dispersion relations are derived by assuming that the Coriolis parameter may be taken to be constant once a scalar equation for a single unknown has been obtained. In the present case this amounts to taking y to be constant in (11), whence the equation admits plane wave solutions of the form

$$v = \exp\{iN(px+qy-\omega t)\},$$

where

$$\omega^3-(p^2+q^2+y^2)\,\omega-N^{-1}p = 0.$$

With a fractional error of order N^{-1} the cubic has approximate solutions

$$\omega = -\frac{N^{-1}p}{p^2+q^2+y^2},$$

the dispersion relation for Rossby waves, and

$$\omega = \pm(p^2+q^2+y^2)^{\frac{1}{2}},$$

the dispersion relation for inertio-gravity waves.

This calculation implies that v should be written as

$$v = v_R(x,y,\tau,N)+v_+(x,y,t,N)+v_-(x,y,t,N),$$

where the notation is obvious. Now the location of the large parameter N in (11) or in the amended form of (11) with τ as time variable indicates that this is a singular perturbation problem, with either the time or the spatial derivatives of v large, of order N. Hence we are led naturally to assuming the asymptotic expansion of v to be of the form

$$v_\pm = v_\pm(x,y,t,N\phi_\pm(x,y,t),N), \tag{19a}$$

$$v_R = v_R(x,y,\tau,N\phi_R(x,y,\tau),N), \tag{19b}$$

where $\phi_\pm$ and ϕ_R are additional unknowns. This prescription is due to Mahony (1962) and is designed to ensure that the asymptotic expansion be uniformly valid in space and time.

The next step would be to substitute (19a) into (11) and (19b) into the amended form of (11), and to solve by expanding $v_\pm$ and v_R in a perturbation series in powers of N^{-1}. It is reasonably obvious that, if this procedure is carried out, the lowest-order equation for either v in (19) involves derivatives only with respect to $N\phi$ and is translationally invariant with respect to this variable. Mahony's advice is to pick ϕ so that, if the solution involves a function of $NF[\phi]$, where F is a functional of ϕ, F is a constant.

These last two sentences lead immediately to equation (15), and we point out that assuming this form of solution is the first step in the geometrical optics approach to wave propagation problems.

In both (17) and (18) we impose the condition that the coefficient of the highest power of N be equated to zero. This leads to

$$\phi_\tau = \frac{\phi_x}{\phi_x^2+\phi_y^2+y^2}, \tag{20}$$

which is obeyed by ϕ_R, and

$$-\phi_t = \pm(\phi_x^2+\phi_y^2+y^2)^{\frac{1}{2}}, \tag{21}$$

which is obeyed by $\phi_\pm$. These will be referred to as dispersion relations but, in contrast to the case of wave propagation in a constant medium, the phase is not a linear function of its arguments.

To satisfy the initial conditions we require that

$$\tilde{\phi}_R = \tilde{\phi}_+ = \tilde{\phi}_- = kx. \tag{22}$$

It can then be shown that

$$\left.\begin{aligned}(\phi_\pm)_t &= \pm(k^2+y^2)^{\frac{1}{2}},\\ (\phi_\pm)_{tt} &= \pm y(k^2+y^2)^{-\frac{1}{2}},\end{aligned}\right\} \tag{23}$$

at $t = 0$, and these equations, together with (12) and (14), imply that at $t = 0$

$$w_R + w_+ + w_- = 0, \tag{24}$$

$$N^{-1}(w_R)_\tau + i(\phi_R)_\tau w_R + (w_+ + w_-)_t = -A_y, \tag{25}$$

and $N^{-2}(w_R)_{\tau\tau} + iN^{-1}[2(\phi_R)_\tau(w_R)_\tau + (\phi_R)_{\tau\tau}w_R] - (\phi_R)_\tau^2 w_R$

$$+ (w_+ + w_-)_{tt} + iN[2(k^2+y^2)^{\frac{1}{2}}(w_- - w_+)_t + y(k^2+y^2)^{-\frac{1}{2}}(w_- - w_+)]$$
$$- N^2(k^2+y^2)(w_+ + w_-) = y(iN^2kA + NA_x). \tag{26}$$

Thus far no approximations have been made. Now, however, we expand each w in an ordinary perturbation series of the form

$$w = w^{(0)} + N^{-1}w^{(1)} + N^{-2}w^{(2)} + \ldots, \tag{27}$$

and substitute into (17), (18) and the initial conditions. It follows that $w_R^{(0)}$ obeys

$$(\phi_x^2 + \phi_y^2 + y^2)w_\tau + (2\phi_x\phi_\tau - 1)w_x + 2\phi_y\phi_\tau w_y + [\phi_\tau\nabla^2\phi + 2\nabla\phi\,.\,\nabla\phi_\tau]w = 0, \tag{28}$$

where ϕ is ϕ_R, and that both of $w_\pm^{(0)}$ satisfy

$$2(\nabla\phi\,.\,\nabla w - \phi_t w_t) + (\nabla^2\phi - \phi_{tt} - i\phi_x/\phi_t)w = 0, \tag{29}$$

where ϕ is $\phi_\pm$. These will be referred to as the transport equations. Substitution of (27) into the initial conditions yields

$$\left.\begin{aligned}\tilde{w}_R^{(0)} + \tilde{w}_+^{(0)} + \tilde{w}_-^{(0)} &= 0,\\ \tilde{w}_+^{(0)} - \tilde{w}_-^{(0)} &= 0,\\ \tilde{w}_+^{(0)} + \tilde{w}_-^{(0)} &= -i\frac{ky}{k^2+y^2}A(x,y),\end{aligned}\right\} \tag{30}$$

whence

$$\tilde{w}_R^{(0)} = \frac{iky}{k^2+y^2}A(x,y), \tag{31a}$$

$$\tilde{w}_+^{(0)} = \tilde{w}_-^{(0)} = -\frac{1}{2}\frac{iky}{k^2+y^2}A(x,y). \tag{31b}$$

We consider first the Rossby mode, and as before omit subscripts and superscripts whenever possible. Our first task is to solve the dispersion relation (20). To this end, let

$$p = \phi_x, \quad q = \phi_y, \quad \sigma = \phi_\tau, \tag{32}$$

so that

$$\sigma(p^2+q^2+y^2) - p = 0. \tag{33}$$

The associated characteristic system of differential equations is (Courant & Hilbert 1962, chapter 2)

$$\left.\begin{aligned}x_\nu = 2p\sigma - 1, \quad y_\nu &= 2q\sigma,\\ \tau_\nu = p^2+q^2+y^2, \quad p_\nu &= 0,\\ q_\nu = -2\sigma y, \quad \sigma_\nu = 0, \quad \phi_\nu &= 2\sigma(p^2+q^2),\end{aligned}\right\} \tag{34}$$

where ν is a parameter. Solving (34) subject to

$$\left.\begin{aligned} x = \lambda, \quad y = \mu, \quad \tau = 0, \quad p = k, \\ q = 0, \quad \sigma = \frac{k}{k^2+y^2}, \quad \phi = kx, \end{aligned}\right\} \tag{35}$$

at $\nu = 0$, where λ and μ are also parameters, we obtain the solution of (20) in parametric form,

$$\phi = \phi_R = k\lambda + \frac{2k^2+\mu^2}{k^2+\mu^2}k\nu - \tfrac{1}{4}\mu^2 \sin\left(\frac{4k\nu}{k^2+\mu^2}\right), \tag{36a}$$

where
$$x = \lambda + \frac{k^2-\mu^2}{k^2+\mu^2}\nu, \quad y = \mu\cos\left(\frac{2k\nu}{k^2+\mu^2}\right), \quad \tau = (k^2+\mu^2)\nu. \tag{36b}$$

We designate the family of curves generated by assigning constant values of λ and μ in (36b) as rays. These are not orthogonal to surfaces of constant phase; instead, as may be verified, they are everywhere tangent to the local vector group velocity of the waves.

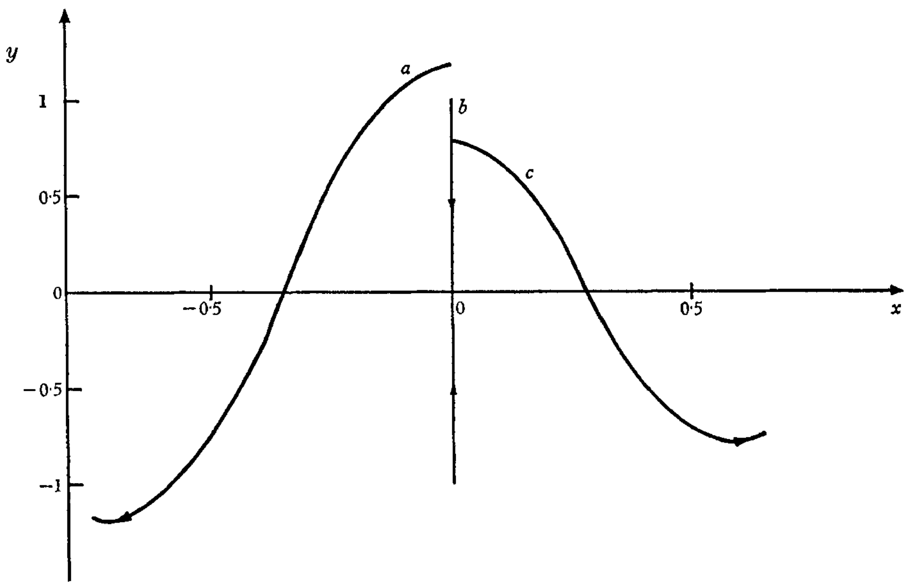

FIGURE 1. Ray paths in (x, y)-plane for Rossby mode, with arrow denoting direction of group velocity. (a) μ = 1·2, k = 1, (b) μ = 1, k = 1, (c) μ = 0·8, k = 1.

We turn now to the problem of solving the transport equation (28). Now from (34),

$$\begin{aligned} \frac{\partial}{\partial\nu} &= x_\nu\frac{\partial}{\partial x} + y_\nu\frac{\partial}{\partial y} + \tau_\nu\frac{\partial}{\partial\tau} \\ &= (2\phi_x\phi_\tau - 1)\frac{\partial}{\partial x} + (2\phi_y\phi_\tau)\frac{\partial}{\partial y} + (\phi_x^2+\phi_y^2+y^2)\frac{\partial}{\partial\tau}, \end{aligned} \tag{37}$$

so (28) is actually an ordinary differential equation, the differentiation in (37) being $\phi_x^2+\phi_y^2+y^2$ times differentiation following points which move with the group velocity. Furthermore, letting

$$J_R = \frac{\partial(x, y, \tau)}{\partial(\lambda, \mu, \nu)} \tag{38}$$

be the Jacobian of the transformation (36*b*), we find after a short calculation that

$$(J_R)_\nu = 2J_R[\phi_\tau \nabla^2\phi + 2\nabla\phi \,.\, \nabla\phi_\tau], \tag{39}$$

so that (28) becomes

$$2J_R w_\nu + (J_R)_\nu w = 0 \tag{40}$$

with solution

$$w = \tilde{w}(\lambda, \mu)\,[\tilde{J}_R(\lambda, \mu)/J_R]^{\frac{1}{2}}. \tag{41}$$

Evaluating the Jacobian and using (31*a*), we obtain

$$w = w_R^{(0)} = \frac{ik\mu}{k^2+\mu^2} A(\lambda, \mu)\left[\cos\left(\frac{2k_\nu}{k^2+\mu^2}\right) + 8\frac{k\mu^2\nu}{(k^2+\mu^2)^2}\sin\left(\frac{2k_\nu}{k^2+\mu^2}\right)\right]^{-\frac{1}{2}}, \tag{42}$$

and the first term in the ray solution for the Rossby mode is

$$v_R^{(0)} = w_R^{(0)} \exp\{iN\phi_R\}.$$

Since the treatment of the + and − gravity modes is identical, we exhibit the calculations only for the former. To solve the dispersion relation, let

$$p = \phi_x, \quad q = \phi_y, \quad \omega = -\phi_t, \tag{43}$$

and we write the dispersion relation as

$$\tfrac{1}{2}(p^2+q^2+y^2-\omega^2) = 0. \tag{44}$$

The characteristic system of differential equations is

$$\left.\begin{aligned} x_s = p, \quad y_s = q, \quad t_s = \omega, \quad p_s = 0,\\ q_s = -y, \quad \omega_s = 0, \quad \phi_s = -y^2, \end{aligned}\right\} \tag{45}$$

where s is a parameter, and these are to be solved subject to

$$\left.\begin{aligned} x = \xi, \quad y = \eta, \quad t = 0, \quad p = k,\\ q = 0, \quad \omega = (k^2+y^2)^{\frac{1}{2}}, \quad \phi = kx, \end{aligned}\right\} \tag{46}$$

at $s = 0$, where ξ and η are also parameters. The solution is

$$\phi = \phi_+ = k\xi - \tfrac{1}{2}\eta^2 s - \tfrac{1}{4}\eta^2 \sin 2s \tag{47a}$$

where

$$x = \xi + ks, \quad y = \eta\cos s, \quad t = (k^2+\eta^2)^{\frac{1}{2}}s, \tag{47b}$$

and as before the rays, generated by assigning constant values to ξ and η in (47*b*), are everywhere tangent to the group velocity.

Since

$$\frac{\partial}{\partial s} = \phi_x\frac{\partial}{\partial x} + \phi_y\frac{\partial}{\partial y} - \phi_t\frac{\partial}{\partial t}, \tag{48}$$

the transport equation (29) is an ordinary differential equation. Also, letting

$$J_+ = \frac{\partial(x, y, t)}{\partial(\xi, \eta, s)} \tag{49}$$

be the Jacobian of the transformation (47*b*), we find that

$$(J_+)_s = J_+(\nabla^2\phi - \phi_{tt}), \tag{50}$$

so that (29) becomes

$$2J_+ w_s + [(J_+)_s + iJ_+ k(k^2+\eta^2)^{-\frac{1}{2}}]w = 0 \tag{51}$$

with solution

$$w = \tilde{w}(\xi,\eta)\,[\tilde{J}_+(\xi,\eta)/J_+]^{\frac{1}{2}}\exp\{-\tfrac{1}{2}ik(k^2+\eta^2)^{-\frac{1}{2}}s\}. \tag{52}$$

Evaluating the Jacobian and using (31 b), we obtain

$$w = w_+^{(0)} = -\frac{1}{2}\frac{ik\eta}{k^2+\eta^2}A(\xi,\eta)\left[\cos s+\frac{\eta^2}{k^2+\eta^2}s\sin s\right]^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}ik(k^2+\eta^2)^{-\frac{1}{2}}s\}$$

and

$$v_+^{(0)} = w_+^{(0)}\exp\{iN\phi_+\}. \tag{53}$$

The solution for $v_-^{(0)}$ is obtained by changing $(k^2+\eta^2)^{\frac{1}{2}}$ to $-(k^2+\eta^2)^{\frac{1}{2}}$ in the above formulae.

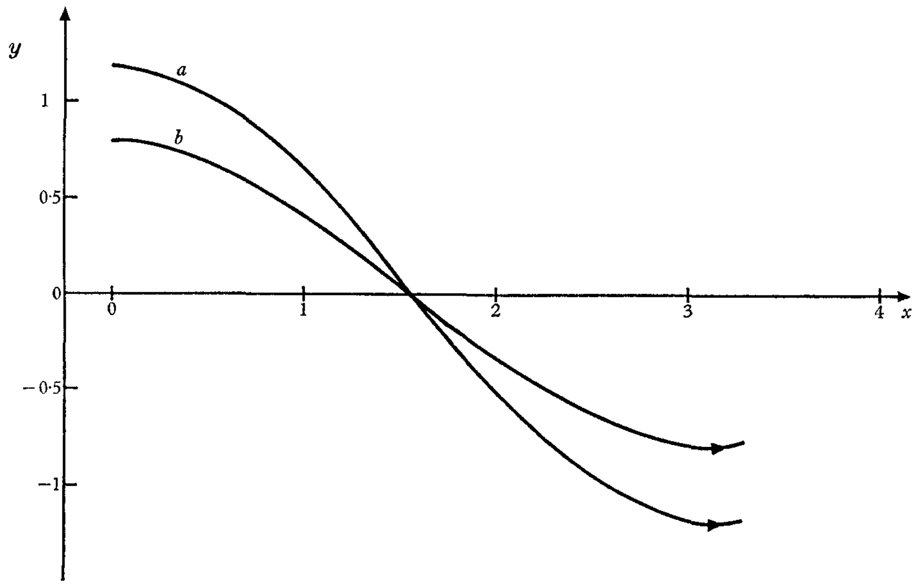

FIGURE 2. Ray paths in (x, y)-plane for + gravity mode. (a) $\eta = 1{\cdot}2$, $k = 1$; (b) $\eta = 0{\cdot}8$, $k = 1$.

The parametric solutions given above are complicated and merit some discussion. Before undertaking this, we give the solutions for u and ζ. For the Rossby mode, let the column vector $\langle u_R, v_R, \zeta_R\rangle$ be written in the form

$$\langle u_R, v_R, \zeta_R\rangle = \psi_R\exp\{iN\phi_R\}, \tag{54}$$

where ψ_R is another column vector. Substituting (54) into (8), (9) and (10), and recalling the definition of τ, we find that

$$\begin{bmatrix} 0 & iy & \phi_x \\ -iy & 0 & \phi_y \\ \phi_x & \phi_y & 0 \end{bmatrix}\psi_R^{(0)} = 0, \tag{55}$$

where ϕ is ϕ_R. The matrix is rank 2 and one can solve for the first and third components of $\psi_R^{(0)}$ in terms of the second. This yields

$$\left.\begin{aligned} u_R^{(0)} &= -\frac{\phi_y}{\phi_x}v_R^{(0)}, \\ \zeta_R^{(0)} &= -\frac{iy}{\phi_x}v_R^{(0)}. \end{aligned}\right\} \tag{56}$$

Similarly, for the gravity waves, we let

$$\langle u_{\pm}, v_{\pm}, \zeta_{\pm}\rangle = \psi_{\pm}\exp\{iN\phi_{\pm}\}, \tag{57}$$

and obtain

$$\begin{bmatrix} \phi_t & iy & \phi_x \\ -iy & \phi_t & \phi_y \\ \phi_x & \phi_y & \phi_t \end{bmatrix} \psi_{\pm}^{(0)} = 0, \tag{58}$$

where ϕ is $\phi_{\pm}$. By virtue of the dispersion relation (21), the matrix in (58) is singular and of rank 2, and $u_{\pm}^{(0)}$ and $\zeta_{\pm}^{(0)}$ are found to be given by

$$\left.\begin{aligned} u_{\pm}^{(0)} &= \frac{\phi_x\phi_t - iy\phi_y}{\phi_y\phi_t + iy\phi_x}\, v_{\pm}^{(0)}, \\ \zeta_{\pm}^{(0)} &= \frac{y^2-\phi_t^2}{\phi_y\phi_t + iy\phi_x}\, v_{\pm}^{(0)}. \end{aligned}\right\} \tag{59}$$

We now turn our attention to the behaviour of the solution in the initial stage of the motion. Now for small τ, the transformation (36*b*) becomes

$$\left.\begin{aligned} x &= \lambda + \frac{k^2-\mu^2}{k^2+\mu^2}\nu, \\ y &= \mu, \\ \tau &= (k^2+\mu^2)\nu, \end{aligned}\right\} \tag{60}$$

with an error of order τ^2, and the Jacobian becomes $k^2+\mu^2$, with a similar error. Thus for small τ

$$v_R^{(0)} = \frac{iky}{k^2+y^2} A\left(x + \frac{y^2-k^2}{(y^2+k^2)^2} N^{-1}t, y\right) \exp\left\{iN\left[kx + \frac{k}{k^2+y^2}N^{-1}t\right]\right\} + O(\tau^2). \tag{61}$$

Similarly, (47*b*) becomes

$$\left.\begin{aligned} x &= \xi + ks, \\ y &= \eta, \\ t &= (k^2+\eta^2)^{\frac{1}{2}} s, \end{aligned}\right\} \tag{62}$$

the Jacobian is $(k^2+\eta^2)^{\frac{1}{2}}$, and

$$\begin{aligned} v_{\pm}^{(0)} = -\frac{1}{2}\frac{iky}{k^2+y^2} A\left(x \mp \frac{kt}{(k^2+y^2)^{\frac{1}{2}}}, y\right), \\ \exp\left\{i\left[N(kx \mp (k^2+y^2)^{\frac{1}{2}}t) - \frac{1}{2}\frac{k}{k^2+y^2}t\right]\right\} + O(t^2). \end{aligned} \tag{63}$$

Equations (61) and (63) state that the energy of each mode travels with the group velocity of that mode. Since the group velocity of the gravity modes is much greater than that of the Rossby mode, the former disperse more rapidly from local sources.

This much could have been anticipated, especially in view of recent work on the concept of group velocity (cf. Landau & Lifshitz 1959, §66). What is surprising is the formation of envelopes of the rays on which the amplitude of the dependent variables is infinite, at least according to the ray approximation. Consider, for example, the solution

$$w = \tilde{w}(\xi,\eta)\,[\tilde{J}_+(\xi,\eta)/J_+]^{\frac{1}{2}}\exp\{-\tfrac{1}{2}ik(k^2+\eta^2)^{-\frac{1}{2}}s\}, \tag{52}$$

$$\phi = k\xi - \tfrac{1}{2}\eta^2 s - \tfrac{1}{4}\eta^2\sin 2s, \tag{47a}$$

where
$$x = \xi + ks, \quad y = \eta\cos s, \quad t = (k^2+\eta^2)^{\frac{1}{2}}s \tag{47b}$$

for the + gravity mode. The Jacobian, given by

$$J_+ = [(k^2+\eta^2)\cos s + \eta^2 s\sin s]/(k^2+\eta^2)^{\frac{1}{2}}, \tag{64}$$

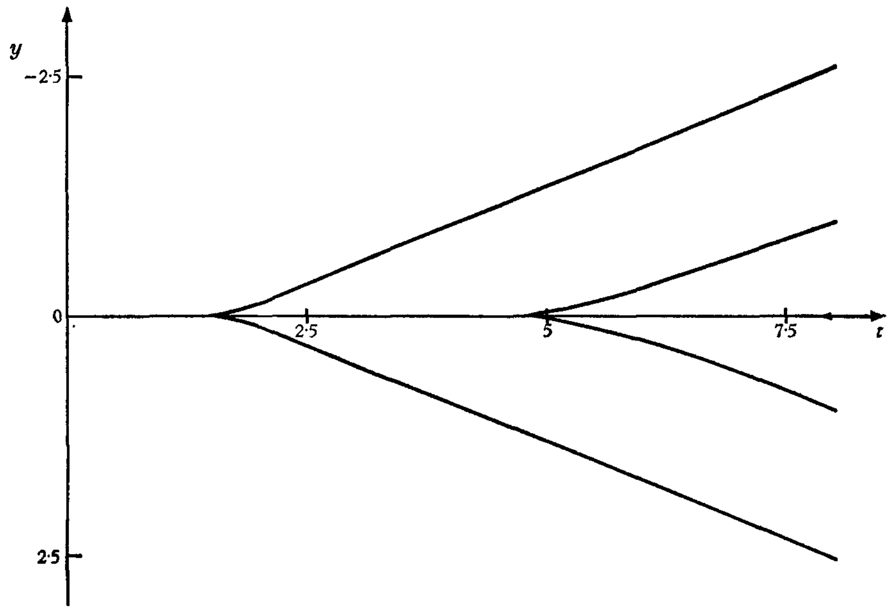

FIGURE 3. Caustics in (y, t)-plane for + gravity mode for $k = 1$.

vanishes for those values of η and s such that

$$\eta^2 = -\frac{k^2\cos s}{s\sin s+\cos s} \tag{65}$$

and consequently for the curves described parametrically by

$$y^2 = -k^2\frac{\cos^3 s}{\cos s + s\sin s}, \quad t^2 = k\left(\frac{s^3\sin s}{s\sin s+\cos s}\right)^{\frac{1}{2}}. \tag{66}$$

On these curves, shown in figure 3, the ray solution for $v_+^{(0)}$ is infinite. Also, the curves are envelopes of the rays, or caustics, and we must anticipate that at least in their neighbourhood the transformation is not one–one. Actually, the situation is far worse.

Consider first the cusps of the caustics, $y = 0$, $t = k(j-\frac{1}{2})\pi$, where j is any positive integer. The pre-images are found by solving

$$0 = \eta\cos s, \quad k(j-\tfrac{1}{2})\pi = (k^2+\eta^2)^{\frac{1}{2}}s. \tag{67}$$

One solution is $\eta = 0$, $s = (j-\frac{1}{2})\pi$. The others are

$$s = (m-\tfrac{1}{2})\pi, \quad \eta^2 = k^2 \frac{(j-\frac{1}{2})^2-(m-\frac{1}{2})^2}{(m-\frac{1}{2})^2}, \tag{68}$$

where m is any integer which is less than j. Consequently, the first cusp has one pre-image, the second three pre-images, the third five pre-images, and so forth.

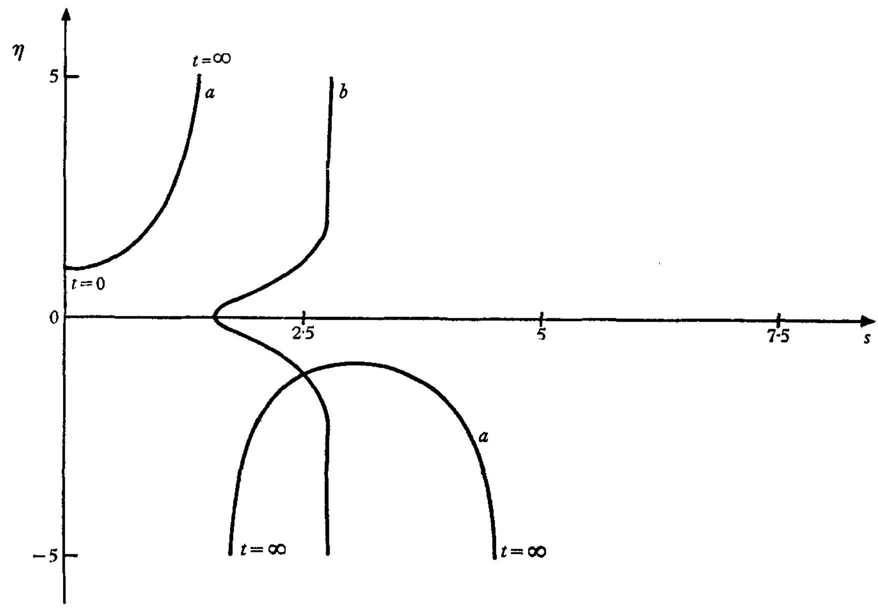

FIGURE 4. Pre-image of curve $y = 1$ for $k = 1$. (*a*) Branches of $y = 1$, (*b*) first caustic.

For considering ordinary points on the caustics and points off the caustics it is useful to plot surfaces of constant y in the (η, s)-plane. Using (47 *b*) and the derived relation

$$\left(\frac{\partial t}{\partial s}\right)_y = \frac{\partial(y,t)}{\partial(y,s)} = \frac{\partial(y,t)}{\partial(\eta,s)}\bigg/\frac{\partial(y,s)}{\partial(\eta,s)} = J_+/\cos s, \tag{69}$$

we find that every ordinary point on the jth caustic has $2j$ pre-images, while every point between the jth and $(j+1)$th caustic has $(2j+1)$ pre-images. The only region for which the transformation is one–one is for those values of t between zero and the first caustic. Elsewhere the solutions are either multi-valued or infinite.

A similar analysis can be made for the other gravity mode and for the Rossby mode, and similar results are obtained. Consequently, further discussion is necessary. This is supplied in part in the next section and in appendix A.

4. Asymptotic solutions near the caustics

The remarks in appendix A indicate that in a region in which solutions are multi-valued the correct solution is obtained by adding the different values. Thus, for example, at any point between the jth and $(j+1)$th caustic for the gravity mode the ray solution has $(2j+1)$ values corresponding to $2j+1$ pre-image points. The correct solution is obtained by summing over all the pre-image points.

In order to carry out this procedure one must assign the correct branch to the square root of $\bar{J}/J$. This is known as finding a phase shift rule. Both the phase shift rule and the correct asymptotic expansion near caustics are obtained by making a local boundary-layer theory. This theory is supplied here, in part, for the gravity modes.

Consider the equation

$$t = s\left(k^2+\frac{y}{\cos^2 s}\right)^{\frac{1}{2}} \equiv F(s,y), \tag{70}$$

which is obtained by eliminating η between the last two equations of (47b). Let overbars denote values on a caustic, and let

$$t = \bar{t}+a, \quad y = \bar{y}+b, \quad s = \bar{s}+\sigma, \quad \eta = \bar{\eta}+\rho, \tag{71}$$

where a, b, σ and ρ are small. We note from (69) that $\bar{F}_s = 0$. Consequently, $\bar{F}_y$ is the slope of the caustic in a (y,t)-plane, and $z = a-\bar{F}_y b$ measures distance from the caustic.

Now from (70) and (71),

$$a = \bar{F}_y b+\tfrac{1}{2}(\bar{F}_{ss}\sigma^2+2\bar{F}_{sy}\sigma b+\bar{F}_{yy}b^2)+\ldots \tag{72}$$

and with the aid of Newton's diagram we find that (72), considered as an equation for σ, has two solutions which tend to zero as a and $b\to 0$. The lowest-order approximation for these is

$$\sigma = \pm\left[\frac{2z}{\bar{F}_{ss}}\right]^{\frac{1}{2}}, \tag{73a}$$

and we similarly find that the corresponding values of ρ are given by

$$\rho = (\bar{\eta}\tan\bar{s})\,\sigma. \tag{73b}$$

Consider the first caustic. Since

$$\bar{F}_{ss} = (\bar{J}_+)_s/\cos\bar{s}$$

is positive and z is positive for $t > \bar{t}$ and negative for $t < \bar{t}$, we see that there are real solutions for σ and ρ for values of y and t on only one side of the caustic. In terms of rays, shown in figure 5, this means that through a point Γ near the caustic in region A there pass two rays which touch the caustic and which coalesce if the point is on the caustic. There is also a 'non-singular' ray through Γ which does not touch the caustic in the neighbourhood of Γ and which need not concern us.

Our starting-point in the boundary-layer analysis is equation (18), which, when the dispersion relation is taken into account, becomes

$$\left(\frac{\partial}{\partial t}+iN\phi_t\right)\{iN[2(\nabla\phi\,.\,\nabla w-\phi_t w_t)+(\nabla^2\phi-\phi_{tt})w] \\ +(\nabla^2 w-w_{tt})\}+Nw_x+iN^2\phi_x w = 0. \tag{74}$$

This equation was solved in §3 by expanding w in a perturbation series in powers of N^{-1} under the assumption that all derivatives are $O(1)$. This assumption is not valid near the caustic; instead we assume that the derivative in the direction normal to the caustic is much larger than the tangential derivatives.

The ensuing analysis is straightforward but very tedious, and we give only the results. Let

$$Z = N^{\frac{2}{3}}(a-\overline{F}_y b) = N^{\frac{2}{3}}z, \tag{75}$$

$$Q = \pm(2/\overline{F}_{ss})^{\frac{1}{2}}, \tag{76}$$

and

$$\delta = 2Q(k^2+\overline{\eta}^2)^{\frac{1}{2}}/\overline{s}. \tag{77}$$

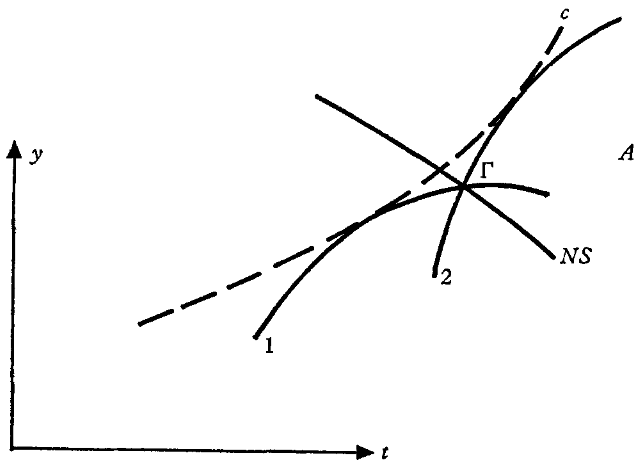

FIGURE 5. Ray paths near a caustic. (1) and (2) are rays which touch the caustic (c), and *NS* is a non-singular ray.

It is found after a standard boundary-layer scaling and much algebra that the boundary-layer version of (74) is

$$w_{ZZ}+i\delta\left(Z^{\frac{1}{2}}w_Z+\frac{w}{4Z^{\frac{1}{2}}}\right) = 0, \tag{78}$$

and that near the caustic

$$N\phi = P+\tfrac{1}{3}\delta Z^{\frac{3}{2}}, \tag{79}$$

where P is a linear function of x, y and t. The solutions for w and ϕ corresponding to the positive value of Q are associated with a ray which has already touched the caustic, while the negative value of Q denotes a ray which has not yet touched the caustic. Letting subscript 1 label the former case and subscript 2 the latter, we find, upon solving (78), that

$$w_{1,2} = \exp\{-\tfrac{1}{3}i\delta Z^{\frac{3}{2}}\}\{A_{1,2}\,\mathrm{Ai}\,[-(\tfrac{1}{2}|\delta|)^{\frac{2}{3}}Z]+B_{1,2}\,\mathrm{Bi}\,[-(\tfrac{1}{2}|\delta|)^{\frac{2}{3}}Z]\}, \tag{80}$$

where $A_{1,2}$ and $B_{1,2}$ are independent of Z and where the standard notation for the Airy functions is used.

Now from (64) and (73), we find that near the caustic $(J_+)_{1,2}$ is given by

$$(J_+)_1 \sim -z, \quad (J_+)_2 \sim +z,$$

as $z \to 0$, with the same proportionality constant. Therefore w_2 as given by the ray approximation is given by

$$w_2 = Sz^{-\frac{1}{4}},$$

as $z \to 0$, where S is independent of z, and we pick A_2 and B_2 so that w_2 as given by (80) matches this as $Z \to +\infty$. Also, we require that

$$v = \exp\{iN\phi_1\}w_1 + \exp\{iN\phi_2\}w_2$$

$\to 0$ and $Z \to -\infty$, since there are no singular rays through points on that side of the caustic with $z < 0$. This determines A_1 and B_1 and hence the solution, which is

$$w_1 = -C\exp\{-\tfrac{1}{3}i\delta Z^{\frac{3}{2}}\}\{\mathrm{Bi}\,(q) + i\,\mathrm{Ai}\,(q)\}, \tag{81a}$$

$$w_2 = C\exp\{-\tfrac{1}{3}i\delta Z^{\frac{3}{2}}\}\{\mathrm{Bi}\,(q) + i\,\mathrm{Ai}\,(q)\}, \tag{81b}$$

where q is the argument of the Airy functions in (80) and where

$$C = S\pi^{\frac{1}{2}}\left(\frac{|\delta|N}{2}\right)^{\frac{1}{6}} e^{\frac{1}{4}i\pi}. \tag{82}$$

This calculation serves two purposes. First, it shows that near the caustic the amplitudes are large, of order $N^{\frac{1}{6}}$. Secondly, since in the limit $Z \to \infty$ (81a) becomes

$$w_1 = e^{-\frac{1}{2}i\pi}S|z|^{-\frac{1}{4}},$$

it provides a phase shift rule, namely that the correct branch of $J_+^{\frac{1}{4}}$ when $J_+ < 0$ is $e^{-\frac{1}{2}i\pi}|J_+|^{\frac{1}{4}}$. A similar calculation for the $-$ gravity mode gives instead a phase shift of $+\frac{1}{2}\pi$, but otherwise the results are identical.

This analysis is incomplete in that it applies only to ordinary points on the caustic. It is conjectured that near the cusps, which lie on the equator, the amplitude is even larger. However, no treatment of this case has been made. Also, the corresponding analysis for the Rossby mode has not been developed, although at least for the ordinary points of the caustics the treatment is probably not difficult.

5. The case of delta function initial data

In this section we consider the problem defined by (11) and the initial conditions

$$\tilde{u} = \tilde{v} = 0, \quad \tilde{\zeta} = \delta(x-x_0)\,\delta(y-y_0), \tag{13}$$

which serves to complement the problem treated in the preceding sections. As before, the ray method will be used.

The major difficulty encountered here is that initially v does not have the form of a wave. In order to treat this case, Lewis (1964) has proposed a heuristic method which has been verified for a number of sample problems. We will give a description of this method which is somewhat different from but equivalent to Lewis's, and at the same time offer a qualitative explanation of why the method works.

The first step is to solve the ray and transport equations for each mode for rays all of which emanate from (x_0, y_0) at time $t = 0$. The solution thus obtained will be non-unique, since neither the initial value of the phase nor that of the amplitude is known. This solution, once the unknown quantities are found, is valid except for t so small that the initial disturbance has not yet formed into a wave-train.

Next, we consider the original equation with new scaled variables $N(x-x_0)$, $N(y-y_0)$, Nt, and solve by expanding v in powers of N^{-1}. The lowest-order equation will have constant coefficients and is equivalent to that obtained by replacing the variable coefficients in the original equation by their values at (x_0, y_0). The solution of the constant coefficient equation is a good approximation to the solution of the original equation except for t so large that secular terms become important.

It is hypothesized that there is an overlapping range of validity in which the small t form of the ray solution agrees with the solution of the constant coefficient problem for moderate or large t. Comparison of the two solutions shows that agreement can be achieved by a proper choice of the unknown quantities appearing in the ray solution. This completes the ray solution, and a uniformly valid expression can be obtained as the sum of the ray and constant coefficient solutions less the form either one of them takes on in the region of overlapping validity. The success of this method is apparently due to the fact that for hyperbolic equations such as (11) the solution depends only on data in a small neighbourhood of (x_0, y_0) if t is small. Hence during the short interval required for a wave-train to form the solution is closely approximated by the solution of the constant coefficient problem.

Turning to the present problem, we consider first the case in which t is sufficiently large for a wave-train to have formed but not so large that the rays have appreciably curved. The solution is approximately that of the constant coefficient problem, which is treated in appendix A. The results contained therein yield, after some algebra,

$$v_R = \frac{N}{4\pi\tau}\Sigma\frac{iay_0}{a^2+b^2+y_0^2}\left|\frac{(a^2+b^2+y_0^2)^5}{y_0^2(3a^2-b^2)-(a^2+b^2)^2}\right|^{\frac{1}{2}}$$

$$\times\exp\left\{i\left[N\frac{2a(a^2+b^2)}{(a^2+b^2+y_0^2)^2}\tau+\tfrac{1}{4}\pi\operatorname{sig}\Phi\right]\right\}, \tag{83}$$

where
$$x = x_0+\frac{a^2-(b^2+y_0^2)}{(a^2+b^2+y_0^2)^2}\tau,\quad y = y_0+\frac{2ab}{(a^2+b^2+y_0^2)^2}\tau, \tag{A 23}$$

and
$$\left.\begin{aligned}\operatorname{sig}\Phi &= 2\operatorname{sgn}a \qquad (a^2+b^2)^2 < y_0^2(3a^2-b^2),\\ &= 0 \qquad\qquad (a^2+b^2)^2 > y_0^2(3a^2-b^2),\end{aligned}\right\} \tag{84}$$

for the Rossby mode, and

$$v_+ = \frac{N}{4\pi t}\,\frac{b(a^2+b^2+y_0^2)^{\frac{1}{2}}-iay_0}{a^2+b^2+y_0^2}\left[\frac{(a^2+b^2+y_0^2)}{y_0}\right]$$

$$\times\exp\left\{i\left[\frac{-Ny_0^2t}{(a^2+b^2+y_0^2)^{\frac{1}{2}}}-\frac{at}{2(a^2+b^2+y_0^2)}+\tfrac{1}{2}\pi\right]\right\}, \tag{85}$$

where
$$x = x_0+\frac{at}{(a^2+b^2+y_0^2)^{\frac{1}{2}}},\quad y = y_0+\frac{bt}{(a^2+b^2+y_0^2)^{\frac{1}{2}}} \tag{A 28}$$

for the + gravity mode. The sum in (83) is over all points (a, b) which are pre-images of a point (x, y, τ).

The Rossby wave solution has been discussed by Longuet-Higgins (1965*a*). The amplitude of the disturbance for this mode is transcendentally small *qua* function of N outside the closed curve

$$y_0^2\frac{y-y_0}{\tau} = \pm 3^{-\frac{3}{2}}\left(1+y_0^2\frac{x-x_0}{\tau}\right)\left(1-8y_0^2\frac{x-x_0}{\tau}\right)^{\frac{1}{2}}, \tag{86}$$

the curve itself being a caustic. The gravity waves spread more rapidly, as can be shown by solving (A 28) to find a and b as functions of x, y and t. Outside the circle $(x-x_0)^2+(y-y_0)^2 = t^2$ the gravity wave solution is small, and the circle is a caustic. At a representative point inside the curve (86) the ratio of v_+ to v_R is that of τ to t, i.e. the amplitude of the gravity mode is smaller than that of the Rossby mode by a factor N^{-1}.

For larger times we need to take into account the curvature of the rays by carrying out the procedure indicated at the start of this section. Turning first to the Rossby mode, we find that the rays are given by

$$\left.\begin{aligned} x &= x_0+\frac{a^2-(b+y_0^2)}{(a^2+b^2+y_0^2)^2}\tau, \\ y &= y_0\cos\left[\frac{2a}{(a^2+b^2+y_0^2)^2}\tau\right]+b\sin\left[\frac{2a}{(a^2+b^2+y_0^2)^2}\tau\right], \end{aligned}\right\} \tag{87}$$

and the phase by

$$\phi = \phi_0+\frac{2a^3}{(a^2+b^2+y_0^2)^2}\tau+\frac{a^2+b^2+y_0^2}{2a}\int_0^\tau (y_\tau)^2\,d\tau. \tag{88}$$

Here ϕ_0 is unknown, and a and b, which serve as parameters, are the values of ϕ_x and ϕ_y at $\tau = 0$. The amplitude which solves the transport equation (28) is

$$w = w_0(a,b)\left|\frac{\partial(x,y)}{\partial(a,b)}\right|^{-\frac{1}{2}}, \tag{89}$$

where w_0 is unknown, and

$$v_R = \Sigma\, w_0(a,b)\left|\frac{\partial(x,y)}{\partial(z,b)}\right|^{-\frac{1}{2}} e^{iN\phi}. \tag{90}$$

Now if (87) and (88) are expanded in powers of τ, it is seen that these expressions agree with (A 23) and (A 25) provided that we take $\phi_0 = 0$. Then, letting

$$w_0 = \frac{N}{2\pi}\frac{iay_0}{a^2+b^2+y_0^2}\exp\{\tfrac{1}{4}i\pi\operatorname{sig}\Phi\}, \tag{91}$$

we obtain agreement between (83) and the small τ form of (90). Hence the ray solution for the Rossby mode is given by (91) with w_0 and ϕ_0 as given above. Similarly, for the + gravity mode, the ray solution which for small t agrees with (85) is

$$v_+ = w_0(a,b)\left|\frac{\partial(x,y)}{\partial(z,b)}\right|^{-\frac{1}{2}}\exp\{-i(at/2(a^2+b^2+y_0^2))\}\exp\{iN\phi\}, \tag{92}$$

where

$$w_0 = \frac{N}{4\pi}\frac{b(a^2+b^2+y_0^2)^{\frac{1}{2}}-iay_0}{a^2+b^2+y_0^2}e^{\frac{1}{2}i\pi}, \tag{93}$$

$$\begin{aligned} x &= x_0+\frac{at}{(a^2+b^2+y_0^2)^{\frac{1}{2}}}, \\ y &= y_0\cos\left[\frac{t}{(a^2+b^2+y_0^2)^{\frac{1}{2}}}\right]+b\sin\left[\frac{t}{(a^2+b^2+y_0^2)^{\frac{1}{2}}}\right] \end{aligned} \tag{94}$$

and

$$\phi = -\int_0^t (a^2+b^2+y_0^2)^{-\frac{1}{2}}\,y^2\,dt. \tag{95}$$

It is easy to prove the existence of caustics due to refraction. Consider, for example, equation (94) for the rays of the + gravity mode. Eliminating a between the two equations, we obtain

$$y/y_0 = \cos(r/(\beta^2+1)^{\frac{1}{2}}) + \beta\sin(r/(\beta^2+1)^{\frac{1}{2}}), \tag{96}$$

where

$$r = [t^2-(x-x_0)^2]^{\frac{1}{2}}/y_0, \quad \beta = b/y_0. \tag{97}$$

The caustics can be found by finding the values of β which solve

$$y_\beta = y_0\{[1+\beta r/(\beta^2+1)^{\frac{3}{2}}\sin(r/(\beta^2+1)^{\frac{1}{2}}) \\ -[\beta^2 r/(\beta^2+1)^{\frac{3}{2}}]\cos(r/(\beta^2+1)^{\frac{1}{2}})\} = 0 \tag{98}$$

and then substituting this value back into (96). One solution is $\beta = \pm\infty$, which corresponds to the circle $(x-x_0)^2+(y-y_0)^2 = t^2$. Another, which is valid when $r = 2n\pi$, n integral, is $\beta = 0$. Hence the point (x, y_0) lies on a caustic whenever

$$(x-x_0)^2 = t^2-4n^2\pi^2y_0^2 \tag{99}$$

and new caustics appear at time intervals of $2n\pi y_0$.

We note finally that the solutions given in this section are not periodic in x but can be used to construct a periodic solution through addition of the solutions for delta functions at $y = y_0$, $x-x_0 = \pm 2m\pi$, m integral. We will not carry out this construction, however.

6. Discussion

The work presented here emphasizes how important it is to take account of the earth's curvature, since it is the curvature which causes the waves to refract and to form envelopes on which the wave amplitude is large. Another effect of the curvature is to produce the phenomenon of equatorial trapping, in which the rays are always confined in a latitude belt around the equator. Bretherton (1964) and Longuet-Higgins (1965*b*) have discussed this effect at length for time-periodic waves.

An important assumption of the present work is that

$$N = 2\Omega R/(gH)^{\frac{1}{2}},$$

the ratio of the radius of the earth to the radius of deformation, is large. The commonly accepted scale height of 8 km for the atmosphere makes $N = 3{\cdot}3$, which is not particularly large. However, this scale height applies to only one of an infinite number of vertical modes, and for the other modes the scale height is smaller and N is larger. The present work is therefore probably qualitatively inaccurate for that mode corresponding to a scale height of 8 km, but accurate for the smaller scale modes.

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Appendix A

This appendix is devoted to solving the model equation

$$\nabla^2 \frac{\partial v}{\partial t} - \frac{\partial^3 v}{\partial t^3} + N\frac{\partial v}{\partial x} - N^2 y_0^2 \frac{\partial v}{\partial t} = 0 \tag{A 1}$$

for $-\infty < x < \infty$, $-\infty < y < \infty$, with initial conditions

$$v = 0, \quad v_t = -F_y, \quad v_{tt} = Ny_0 F_x, \tag{A 2}$$

where $F(x,y)$ is a known function, y_0 is constant, and $N \gg 1$. The calculation provides information necessary for solving the initial-value problem (11) for non-oscillatory initial conditions and in addition serves to resolve certain difficulties associated with the ray solution.

To solve (A 1) we introduce the Fourier transform

$$\bar{v} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v \exp\{iN(ax+by)\}\,dx\,dy. \tag{A 3}$$

The transform of (A 1) is

$$\bar{v}_{ttt} + N^2(a^2+b^2+y_0^2)\,\bar{v}_t - iN^2 a\bar{v} = 0 \tag{A 4}$$

and has as its solution

$$\bar{v} = \bar{F}\sum_{\alpha=1}^{3} c_\alpha \exp\{-iN\omega_\alpha t\}, \tag{A 5}$$

where the ω_α's and c_α's solve

$$\omega^3 - (a^2+b^2+y_0^2)\,\omega - N^{-1}a = 0 \tag{A 6}$$

and

$$\sum_\alpha c_\alpha = 0, \quad \sum_\alpha \omega_\alpha c_\alpha = b, \quad \sum_\alpha \omega_\alpha^2 c_\alpha = -iay_0, \tag{A 7}$$

respectively. It is readily verified that the solution of (A 6) is

$$\left.\begin{aligned} \omega_1 &= -\frac{aN^{-1}}{a^2+b^2+y_0^2} + O(N^{-3}) \equiv \sigma_1 + O(N^{-3}),\\ \omega_{\substack{2\\3}} &= \pm(a^2+b^2+y_0^2)^{\frac{1}{2}} + \frac{aN^{-1}}{2(a^2+b^2+y_0^2)} + O(N^{-2})\\ &\equiv \sigma_{\substack{2\\3}} + \frac{aN^{-1}}{2(a^2+b^2+y_0^2)} + O(N^{-2}) \end{aligned}\right\} \tag{A 8}$$

and that, with an error of order N^{-1},

$$\left.\begin{aligned} c_1 &= iay_0/(a^2+b^2+y_0^2),\\ c_{\substack{2\\3}} &= [\pm b(a^2+b^2+y_0^2)^{\frac{1}{2}} - iay_0]/2(a^2+b^2+y_0^2). \end{aligned}\right\} \tag{A 9}$$

Letting

$$d_1 = c_1, \quad d_{\substack{2\\3}} = c_{\substack{2\\3}}\exp\{-i(at/2(a^2+b^2+y_0^2))\}, \tag{A 10}$$

we obtain the solution of (A 1) in the form

$$\begin{aligned} v &= \left(\frac{N}{2\pi}\right)^2 \sum_\alpha \iint \bar{F} d_\alpha \exp\{iN(ax+by-\sigma_\alpha t)\}\,da\,db\\ &\equiv \sum_\alpha v_\alpha. \end{aligned} \tag{A 11}$$

The integrals occurring in (A 11) will now be evaluated through use of Lewis's (1964) result that if

$$I = \int g(t_1, \ldots, t_n) \exp\{iN\phi(t_1, \ldots, t_n)\} dt_1 \ldots dt_n,$$

then, as $N \to \infty$,

$$I \sim \left(\frac{2\pi}{N}\right)^{\frac{1}{2}n} \Sigma |\det \Phi|^{-\frac{1}{2}} g \exp\{i[N\phi + \tfrac{1}{4}\pi \operatorname{sig} \Phi]\},$$

where the sum is over all points $(t_1, \ldots, t_n)$ such that

$$\frac{\partial \phi}{\partial t_i} = 0,$$

Φ is the matrix

$$\Phi = \frac{\partial^2 \phi}{\partial t_i \, \partial t_j},$$

and sig Φ is the sum of the signs of the eigenvalues of Φ.

We consider first oscillatory initial data,

$$F = A(x, y)\, e^{iNkx}. \tag{A 12}$$

Then

$$v_1 = \left(\frac{N}{2\pi}\right)^2 \iiiint d_1(a, b) A(\xi, \eta)\, e^{iN\phi} da\, db\, d\xi\, d\eta, \tag{A 13}$$

where

$$\phi = k\xi + a(x - \xi) + b(y - \eta) + \frac{a}{a^2 + b^2 + y_0^2}\tau, \tag{A 14}$$

and $\tau = N^{-1}t$ as before. The points of stationary phase satisfy

$$a = k, \quad b = 0, \quad x = \xi + \frac{k^2 - y_0^2}{(k^2 + y_0^2)^2}\tau, \quad y = \eta, \tag{A 15}$$

and for these points

$$|\det \Phi| = 1, \quad \operatorname{sig} \Phi = 0, \quad \phi = kx + \frac{k}{k^2 + y_0^2}\tau. \tag{A 16}$$

It follows that

$$v_1 \sim d_1(k, 0) A(\xi, \eta) \exp\left\{iN\left[kx + \frac{k}{k^2 + y_0^2}\tau\right]\right\}, \tag{A 17}$$

which except for y_0 in place of y is identical to (61), the small τ form of the Rossby mode. Similarly, for v_2,

$$a = k, \quad b = 0, \quad x = \xi + \frac{kt}{(k^2 + y_0^2)^{\frac{1}{2}}}, \quad y = \eta, \tag{A 18}$$

and we obtain

$$v_2 \sim d_2(k, 0) A(\xi, \eta) \exp\{i[N(kx - (k^2 + y_0^2)^{\frac{1}{2}} t]\}, \tag{A 19}$$

which bears the same relation to (62) as (A 17) does to (61). In addition, it can be shown that (A 17) and (A 19) agree with the solution of (A 1) obtained through use of the ray method.

For the initial data

$$F = \delta(x - x_0)\, \delta(y - y_0), \tag{A 20}$$

$$v_1 = \left(\frac{N}{2\pi}\right)^2 \iint d_1(a, b)\, e^{iN\phi} da\, db, \tag{A 21}$$

where

$$\phi = a(x - x_0) + b(y - y_0) + \frac{a}{a^2 + b^2 + y_0^2}\tau. \tag{A 22}$$

The sum in the stationary phase formula is over those points (a,b) such that

$$x = x_0 + \frac{a^2-(b^2+y_0^2)}{(a^2+b^2+y_0^2)^2}\tau, \quad y = y_0 + \frac{2ab}{(a^2+b^2+y_0^2)^2}\tau, \tag{A 23}$$

$|\det \Phi|$ is given by

$$|\det \Phi| = \left\|\begin{matrix}\phi_{aa} & \phi_{ab}\\ \phi_{ab} & \phi_{bb}\end{matrix}\right\| = \left\|\begin{matrix}-x_a & -x_b\\ -y_a & -y_b\end{matrix}\right\|$$

$$= \left|\frac{\partial(x,y)}{\partial(a,b)}\right|, \tag{A 24}$$

$$\phi = \frac{2a(a^2+b^2)}{(a^2+b^2+y_0^2)^2}\tau \tag{A 25}$$

and $$v_1 \sim \frac{N}{2\pi}\Sigma\, d_1(a,b)\left|\frac{\partial(x,y)}{\partial(a,b)}\right|^{-\frac{1}{2}}\exp\left\{i\left[\frac{2a(a^2+b^2)}{(a^2+b^2+y_0^2)^2}N\tau + \tfrac{1}{4}\pi\,\mathrm{sig}\,\Phi\right]\right\}, \tag{A 26}$$

a result needed in §5. Similarly, for the + gravity mode,

$$v_2 \sim \frac{N}{2\pi}d_2(a,b)\left|\frac{\partial(x,y)}{\partial(a,b)}\right|^{-\frac{1}{2}}\exp\left\{i\left[-\frac{Ny_0^2t}{(a^2+b^2+y_0^2)^{\frac{1}{2}}} + \tfrac{1}{4}\pi\,\mathrm{sig}\,\Phi\right]\right\}, \tag{A 27}$$

where $$x = x_0 + \frac{at}{(a^2+b^2+y_0^2)^{\frac{1}{2}}}, \quad y = y_0 + \frac{bt}{(a^2+b^2+y_0^2)^{\frac{1}{2}}}. \tag{A 28}$$

We note that, for both types of initial conditions, the asymptotic expansion of the solution is of the same form, a 'wave' with variable amplitude and rapidly varying phase. The solution is parametric, and the relations between the physical co-ordinates and the parameters may be thought of as defining curves in space and time, the rays. The sum in the stationary phase formula is over those rays which pass through a point (x,y,t). On envelopes of rays the stationary phase result is invalid, and another asymptotic evaluation must be used, that of Chester *et al.* (1957).

The solution of constant coefficient problems by Fourier analysis and asymptotic integration is useful for interpreting the ray method of solution. Besides motivating the form of solution, it indicates that when more than one ray passes through a point the solution is the sum of the solutions associated with each ray. Furthermore, it indicates that a different type of asymptotic expansion, a local boundary-layer theory, is needed to deal with envelopes, and it provides a check on the boundary-layer theory. Calculations by Buchal & Keller (1960) and by the author do in fact indicate that, when such a check is possible, the boundary-layer theory yields the same results as those given by the method of Chester *et al.*

A great advantage of the ray method is that it can be used to solve equations with non-constant coefficients provided that the scale over which the coefficients vary significantly is much greater than the scale of the dependent variables. An alternate procedure is to use the closely related method of multiple scales (cf. Carrier 1966). However, the ray method is more systematic and lends itself more readily to obtaining higher approximations.

Appendix B

The calculations here have been carried out under the β-plane approximation, and it is natural to inquire what changes in the previous results occur when this approximation is not made. The dispersion relations, at least, are easily obtained, and are similar to those obtained under the β-plane approximation. The method used is due to Lax (1957).

Let u and v be the non-dimensional velocity components, and let

$$u = mU, \quad v = mV, \tag{B 1}$$

and recall that

$$f = \sin\phi = \tanh y, \quad m = \sec\phi = \cosh y. \tag{5}$$

Then the non-dimensional equations of motion in Mercator co-ordinates are

$$\frac{\partial U}{\partial t} - NfV + \frac{\partial \zeta}{\partial x} = 0, \tag{B 2}$$

$$\frac{\partial V}{\partial t} + NfU + \frac{\partial \zeta}{\partial y} = 0, \tag{B 3}$$

$$\frac{\partial \zeta}{\partial t} + m^2\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) = 0. \tag{B 4}$$

As before, we assume that solutions may be split into a Rossby mode with time variable $\tau = N^{-1}t$ and two gravity modes. For the Rossby mode, let

$$\langle U_R, V_R, \zeta_R\rangle = \psi_R \exp\{iN\phi_R\}, \tag{B 5}$$

where as before the vectors are column vectors, and substitute into (B 2) to (B 4). This yields

$$iA\psi_R + N^{-1}D\psi_R + N^{-2}(\psi_R)_\tau = 0, \tag{B 6}$$

where

$$A = \begin{bmatrix} 0 & if & \phi_x \\ -if & 0 & \phi_y \\ m^2\phi_x & m^2\phi_y & 0 \end{bmatrix}, \quad D = \begin{bmatrix} \phi_\tau & 0 & \dfrac{\partial}{\partial x} \\ 0 & \phi_\tau & \dfrac{\partial}{\partial y} \\ m^2\dfrac{\partial}{\partial x} & m^2\dfrac{\partial}{\partial y} & \phi_\tau \end{bmatrix}, \tag{B 7}$$

and ϕ is ϕ_R. Expanding ψ_R in powers of N^{-1}, we obtain

$$A\psi_R^{(0)} = 0, \quad iA\psi_R^{(1)} + D\psi_R^{(0)} = 0, \tag{B 8}$$

and higher-order equations.

The matrix A is of rank 2 and has right and left null vectors such that

$$lA = Ar = 0, \tag{B 9}$$

where l is a row vector and r a column vector. These are uniquely determined except for a multiplicative constant, and a convenient choice is

$$l = (m^2\phi_y, -m^2\phi_x, -if), \quad r = \langle\phi_y, -\phi_x, if\rangle. \tag{B 10}$$

From the first equation in (B 8) $\psi_R^{(0)}$ is a scalar multiple of r,

$$\psi_R^{(0)} = \theta_R^{(0)} r; \tag{B 11}$$

and, if we substitute this into the second equation in (B 8) and multiply by l, we obtain

$$lD\theta_R^{(0)} r = 0, \tag{B 12}$$

which after the necessary algebra is

$$\{\phi_\tau[m^2(\phi_x^2+\phi_y^2)+f^2]-\phi_x\}\theta_R^{(0)} = 0. \tag{B13}$$

Hence the dispersion relation obeyed by ϕ_R is

$$\phi_\tau = \frac{\phi_x}{[m^2(\phi_x^2+\phi_y^2)+f^2]}. \tag{B14}$$

The dispersion relation for the gravity modes is obtained much more simply. Let

$$\langle U_\pm, V_\pm, \zeta_\pm\rangle = \psi_\pm \exp\{iN\phi_\pm\}, \tag{B15}$$

substitute into the equations of motion and expand $\psi_\pm$ in powers of N^{-1}. The lowest-order equation is

$$\begin{bmatrix}\phi_t & if & \phi_x\\ -if & \phi_t & \phi_y\\ m^2\phi_x & m^2\phi_y & \phi_t\end{bmatrix}\psi_\pm^{(0)} = 0, \tag{B16}$$

where ϕ is $\phi_\pm$, and for $\psi_\pm^{(0)}$ not to be zero the matrix must be singular. Its determinant is

$$\phi_t[\phi_t^2-f^2-m^2(\phi_x^2+\phi_y^2)],$$

and the case $\phi_t = 0$ has already been incorporated in the Rossby mode. Consequently, $\phi_\pm$ obeys

$$-\phi_{t\pm} = \pm[m^2(\phi_x^2+\phi_y^2)+f^2]^{\frac{1}{2}}. \tag{B17}$$

The ray paths for the different modes can be obtained by quadrature, and are qualitatively similar to those obtained through use of the β-plane approximation.

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